Supplemental Material:
Production Function Estimation with
Unobserved Input Price Dispersion*

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Online Appendix 1:   The General Parametric Model

This appendix illustrates our approach for general forms of production. We provide conditions under which unobserved materials inputs can be recovered from a parametric specification of the production function.

1.1   Model Setup

Suppose at each period t, each firm j produces a single product using labor (L_{jt}), intermediate material (M_{jt}), and capital (K_{jt}) via the production function,

$$Q_{jt} = e^{\omega_{jt}} F(L_{jt}, M_{jt}, K_{jt}; \theta),$$

where Q_{jt} is the output quantity, ω_{jt} is a Hicks-neutral productivity shock observed by the firm (but not by researchers), and θ is the set of parameters in the production function. The inverse demand function is,

$$P_{jt} = P_t(Q_{jt}; \eta),$$

where P_{jt} is the output price and η is the set of parameters in the demand function. We allow the demand function to be different over time. We first restate the assumptions from the main paper:

Assumption 1 (Exogenous Input Prices). *Firms are price takers in input markets. Suppliers use linear pricing, but input prices are allowed to be different across firms and over time. Prices have strictly positive support.*

Assumption 2 (Profit Maximization). *After observing their productivity draw, ω_{jt}, and firm-specific input prices, firms optimally choose labor and material inputs to maximize the profit in each period. The firm's capital stock for period t is chosen prior to the revelation of ω_{jt}.*

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Assumption 3 (Data). *The researcher observes revenue $R_{jt} = P_{jt}Q_{jt}$, inputs expenditure $E_{Mjt} = P_{M_{jt}}M_{jt}$, wage rate $P_{L_{jt}}$, number of workers or number of working hours L_{jt}, and capital stock K_{jt}. But she does not observe the prices and quantities of either outputs (i.e., P_{jt} and Q_{jt}), materials inputs (i.e., $P_{M_{jt}}$ and M_{jt}), or productivity ω_{jt}. All these variables are observed (or chosen) by the firm.*

We now add two additional assumptions that restrict the functional form of the production and inverse demand functions.

Assumption 4 (Production Function). *Production function $F(\cdot)$ is known up to a finite dimensional parameter θ, strictly increasing in inputs, and continuously differentiable up to second order. The limits as inputs go to infinity of the marginal product of labor and materials exist.*

Assumption 5 (Inverse Demand Function). *The inverse demand function is continuous, decreasing, differentiable and satisfies $\lim_{Q \to \infty} P_t(Q; \eta) = 0$.*

Assumptions 4 and 5 are standard, although sometimes implicit, in the literature. They ensure that we can use first order conditions and that a solution to the firm's maximization problem exists. Assumption 1 is our primary departure from the earlier literature, it weakens the typical assumption that input prices are homogeneous when they are not observed. The assumption that firms are price takers does not preclude them being offered different prices on the basis of their size (i.e., capital stock), productivity, or negotiating ability, but does assume that firms do not receive "quantity discounts," which would endogenously affect purchasing decisions.

1.2 Recovering Materials Quantities

Given our assumptions, a finite solution to the profit maximization problem exists,[1] and we focus on interior solutions where the first order conditions hold.[2] Following the main paper, we manipulate the ratio of the first order conditions with respect to labor and materials to arrive at the following equality:

$$\frac{F_{L_{jt}}L_{jt}}{F_{M_{jt}}M_{jt}} = \frac{E_{L_{jt}}}{E_{M_{jt}}} \tag{1}$$

where $E_{L_{jt}} = P_{L_{jt}}L_{jt}$ and $E_{M_{jt}} = P_{M_{jt}}M_{jt}$. In the main paper, we showed that for the CES production function this equality yielded a closed form solution for materials inputs. The following proposition provides general conditions under which (1) admits a unique solution.

Proposition 1. *Define*

$$z(M_{jt}; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) \;\; \equiv \;\; \frac{F_{L_{jt}}L_{jt}}{F_{M_{jt}}M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}}.$$

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[1]To see this, notice that under Assumption 4 and 5, the marginal return to materials (and labor) must eventually be driven below the price of materials. If the production function asymptotes to a finite level of production, then the marginal returns of inputs go to zero as L_{jt} and M_{jt} go to infinity. Alternatively, if Q_{jt} goes to infinity as L_{jt} and M_{jt} rise, then the left hand side of is weakly negative in the limit, so the marginal cost of inputs must outweigh the marginal return.

[2]Corner solutions where $L_{jt} = 0$ or $M_{jt} = 0$ are readily observable in the data. We observe L_{jt} directly, and know that $M_{jt} = 0$ whenever expenditure on materials is zero, since prices are assumed to be positive. Zero inputs expenditure does not occur in our data, and we expect it to be rare in most datasets. Corner solutions can simply be dropped when using our method, although observing zero expenditure for intermediate inputs may lead one to be concerned about severe measurement error.

For a given point $(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt})$ *generated from firm profit maximization and parameter vector θ, suppose either (1)* $\frac{\partial z}{\partial M} > 0$ *for all $M \in (0, \infty)$ such that $z(M; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0$, or (2)* $\frac{\partial z}{\partial M} < 0$ *for all $M \in (0, \infty)$ such that $z(M; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0$. Then there exists a unique M^* that satisfies,*

$$z(M^*; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0.$$

Proof. Recall that we assume the first order conditions of profit maximization hold, which implies the existence of a solution for $z(\cdot)$. We show the uniqueness of the solution by contradiction. Suppose that there are multiple M in the set $\mathcal{M} = \{M : z(M, L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0\}$, and for all of them $\frac{\partial z}{\partial M} > 0$. Take any two consecutive solutions $M', M'' \in \mathcal{M}$ such that M'' is the smallest member of $\mathcal{M}$ that is larger than M'. Take a sequence converging to M' from above, we know there exist some m' such that $z(m', L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) > 0$. Likewise, take a sequence converging to M'' from below, we know there exist some m'' such that $z(m', L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) < 0$. Because $z(\cdot)$ is continuous in M, which is guaranteed by Assumption 4, there must be some $m^* \in (m', m'')$ such that $z(m^*, L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0$. However, this contradicts M' and M'' being consecutive members of $\mathcal{M}$. $\square$

The idea of the proof is that if z satisfies a single crossing property in M_{jt}, then we can always uniquely recover the material quantity from observed data. Proposition 1 provides a sufficient condition for single crossing to hold. The condition of the proof is sufficient, but not necessary, as it does not account for the unique zero occurring at an inflection point. However, this is a knife-edged case. Proposition 1 can also be applied to show that M^* can be recovered for a specific functional form, as we will see in the CES and translog examples below. Of course, for some specifications of the production function, it is possible the condition will not hold and multiple materials quantity-price combinations may satisfy (1). In this case, the model may be partially identified. For the remainder of this paper, we will assume the conditions in Proposition 1 hold so that M^* can be uniquely recovered. Once we recover $M_{jt}^* = M^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta)$, we can replace the unobserved intermediate inputs M_{jt} in the first order condition for labor to back out the productivity shock as $\omega_{jt}^* = \omega^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta, \eta)$.[3]

The following example illustrates a case where the conditions of Proposition 1 are *not* satisfied for the Cobb-Douglas specification. This is because, as is well-known, the Cobb-Douglas production function assumes that expenditure shares are constant within the data, eliminating the source of variation we need to separate input prices and quantities. The case of the CES production function is covered in the main paper. Finally, we consider the case of the translog production function in Online Appendix 2:

Example 1 (Cobb-Douglas Production Function)

For Cobb-Douglas production function

$$Q_{jt} = e^{\omega_{jt}} F(L_{jt}, M_{jt}, K_{jt}; \theta) = e^{\omega_{jt}} L_{jt}^{\alpha_L} M_{jt}^{\alpha_M} K_{jt}^{\alpha_K},$$

---

[3]Note that we make use of the expenditure identities and the production function itself to solve the first order conditions as a function of these variables.

where $\theta = (\alpha_L, \alpha_M, \alpha_K)$, it is straightforward to show that,

$$
\begin{aligned}
z(M_{jt}; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) &= \frac{F_{L_{jt}} L_{jt}}{F_{M_{jt}} M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{\alpha_L L_{jt}^{\alpha_L - 1} M_{jt}^{\alpha_M} K_{jt}^{\alpha_K} L_{jt}}{\alpha_M L_{jt}^{\alpha_L} M_{jt}^{\alpha_M - 1} K_{jt}^{\alpha_K} M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{\alpha_L}{\alpha_M} - \frac{E_{L_{jt}}}{E_{M_{jt}}}.
\end{aligned}
$$

In this case, $z(\cdot)$ does not vary with M_{jt} (e.g., $\frac{\partial z}{\partial M_{jt}} = 0$), so unobserved materials cannot be recovered from (1). The intuition is that, because the elasticity of substitution is fixed at one, when the relative inputs price ($\frac{P_L}{P_M}$) changes firms always choose labor and material such that the percentage increase (or decrease) of the labor-material ratio ($\frac{L}{M}$) equals the percentage decrease (or increase) of the relative price ($\frac{P_L}{P_M}$). As a result, the expenditure ratio $\frac{E_{L_{jt}}}{E_{M_{jt}}}$ remains constant ($\frac{\alpha_L}{\alpha_M}$). In this case, we cannot separate the price and quantity of materials from the information on the expenditure ratio $\frac{E_{L_{jt}}}{E_{M_{jt}}}$. ∎

Example 2 (CES Production Function)

Consider CES production function,[4]

$$
Q_{jt} = e^{\omega_{jt}} F(L_{jt}, M_{jt}, K_{jt}; \theta) = e^{\omega_{jt}} [\alpha_L L_{jt}^{\gamma} + \alpha_M M_{jt}^{\gamma} + \alpha_K K_{jt}^{\gamma}]^{\frac{1}{\gamma}}, \tag{2}
$$

where $\gamma = \frac{\sigma - 1}{\sigma}$ (σ is the elasticity of substitution), and $\theta = (\alpha_L, \alpha_M, \alpha_K, \sigma)$. We can show that,

$$
\begin{aligned}
z(M_{jt}; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) &= \frac{F_{L_{jt}} L_{jt}}{F_{M_{jt}} M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{e^{\omega_{jt}} [\alpha_L L_{jt}^{\gamma} + \alpha_M M_{jt}^{\gamma} + \alpha_K K_{jt}^{\gamma}]^{\frac{1}{\gamma} - 1} \alpha_L L_{jt}^{\gamma - 1} L_{jt}}{e^{\omega_{jt}} [\alpha_L L_{jt}^{\gamma} + \alpha_M M_{jt}^{\gamma} + \alpha_K K_{jt}^{\gamma}]^{\frac{1}{\gamma} - 1} \alpha_M M_{jt}^{\gamma - 1} M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{\alpha_L L_{jt}^{\gamma}}{\alpha_M M_{jt}^{\gamma}} - \frac{E_{L_{jt}}}{E_{M_{jt}}}
\end{aligned}
$$

Taking the derivative of $z(\cdot)$ with respect to M_{jt} we yields,

$$
\frac{\partial z}{\partial M_{jt}} = -\gamma \frac{\alpha_L L_{jt}^{\gamma}}{\alpha_M M_{jt}^{\gamma + 1}},
$$

It is clear that the sign of $\frac{\partial z}{\partial M_{jt}}$ is determined by γ only.[5] Therefore, as long as $\gamma \neq 0$ (i.e., $\sigma \neq 1$), Proposition 1 is satisfied and we can recover the unobserved material from (1). This yields the

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[4]In the main paper we fully explore a normalized form of the CES production function, but we use an unnormalized form here for expositional simplicity.

[5]When $\gamma = 0$, the CES function is equivalent to the Cobb-Douglas case (Example 3.1) and we cannot recover the unobserved materials from (1). This case must be excluded from the parameter set Θ.

closed form,

$$M^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta) = \left(\frac{\alpha_L}{\alpha_M} \frac{E_{Mjt}}{E_{Ljt}} \right)^{\frac{1}{\gamma}} L_{jt}.$$

∎

1.3 Estimation

We now turn to estimation of the parameters of the production function, θ, and the inverse demand function, η. If our assumptions are satisfied, the intermediate input quantity can be uniquely recovered as,

$$M_{jt}^* = M^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta). \tag{3}$$

Then we can plug M_{jt}^* back into the first order conditions and recover the unobserved productivity,

$$\omega_{jt}^* = \omega^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta, \eta). \tag{4}$$

Different from Olley and Pakes (1996), here we recover the unobserved productivity parametrically from the firm's first order conditions.[6] There are several advantages to this method. First, the estimation does not require investment data. There is no need to rely on invertibility of the investment policy function, which may be problematic when adjustment costs generate lumpiness in the optimal investment policy. Moreover, our method of controlling for endogeneity does not require the Markov assumption on the productivity evolution process.[7] Finally, we fully exploit the structural assumptions of the parametric production function and corresponding first order conditions to recover unobserved productivity and material quantity—as in Doraszelski and Jaumandreu (2013)—so we do not have to rely on nonparametric methods to estimate these functions. As a result, even when both labor and material expenditures may be functions of the same set of variables in (5) we still have identification as long as labor and materials expenditures are not perfectly correlated.[8] Ackerberg et al. (2006) also discussed this possibility (page 16, version of December 28, 2006).

Since output quantities are not directly observed, we follow Klette and Griliches (1996) and use the revenue function as the estimation equation. The revenue function is,

$$R_{jt} = e^{u_{jt}} P_t (Q_{jt}; \eta) Q_{jt}.$$

Where R_{jt} is the observed revenue of the firm, $Q_{jt} = e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta)$ is the predicted quantity of physical output based on observed inputs and the model parameters (θ, η), and u_{jt} is a mean-zero revenue error term which incorporates measurement error as well as demand and productivity shocks that are unanticipated by the firm. Taking the logarithm of the revenue function yields,

$$\ln R_{jt} = \ln P_t \left(e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta); \eta \right) + \ln \left[e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta) \right] + u_{jt}. \tag{5}$$

---

[6]Zhang (2014) uses a similar approach to allow for biased technical change in the production function.

[7]Of course, assumptions on the productivity evolution process may still be needed in identifying the production function, as we will discuss below.

[8]The data in our application shows that the labor-material expenditure ratio has large variation across firms (as required by our empirical model), supporting the idea that these two expenditures are not perfectly correlated.

In this equation, the unobserved productivity and material quantity, ω_{jt}^* and M_{jt}^*, are functions of observed variables as in equations (3) and (4). The only remaining unobservable, u_{jt}, is unknown to the firm and is uncorrelated with the observed inputs.

To simplify notations, denote $\mathbf{w}_{jt} \equiv (L_{jt}, E_{M_{jt}}, E_{L_{jt}}, K_{jt})$, $r_{jt} \equiv \ln R_{jt}$, and $\beta \equiv (\theta, \eta) \in \mathbb{R}^D$. Define

$$f(\mathbf{w}_{jt}; \beta) = \ln P_t \left(e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta); \eta \right) + \ln \left[e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta) \right].$$

Therefore, the true parameter β_0 solves the following nonlinear least squares problem,

$$\min_{\beta} E \left[(r_{jt} - f(\mathbf{w}_{jt}; \beta))^2 \right]. \tag{6}$$

Of course, we have not yet shown that β_0 is identified. Indeed, in both of our primary examples we need additional restrictions to identify β_0. In order to accommodate these additional restrictions, we cast the non-linear least square problem in terms of the generalized method of moments (GMM) via its first order conditions. To be specific, the first order conditions of the non-linear least squares (6) are,

$$E \left[\nabla_\beta f(\mathbf{w}_{jt}; \beta) \left(r_{jt} - f(\mathbf{w}_{jt}; \beta) \right) \right] = 0,$$

where $\nabla_\beta f(\mathbf{w}_{jt}; \beta)$ is the $D \times 1$ vector of partial derivatives with respective to β.

The GMM framework allows us to easily add additional restrictions in a manner similar to Wooldridge (2009). For the CES, these restrictions are related to aggregate measures and do not involve any additional assumptions. For the translog, we rely on the additional assumption that productivity moves according to a Markov process to provide moment restrictions with which to identify all the parameters. This second approach is quite general and can provide identifying restrictions for many functional forms (including the CES, if it were necessary). In both cases, these restrictions can be imposed in terms of moment conditions:

$$E[\mathbf{h}(\mathbf{x}_{jt}; \beta)] = 0,$$

where $\mathbf{h}(\mathbf{x}_{jt}; \beta)$ is a $S \times 1$ dimension function regarding observable exogenous variables $\mathbf{x}_{jt}$ (which may include $\mathbf{w}_{jt}$) and the parameter vector β. Define $\Phi(\beta)$ as a $D \times D$ matrix,

$$\Phi(\beta) = E \left[\left(\nabla_\beta f(\mathbf{w}_{jt}; \beta) \right) \left(\nabla_\beta f(\mathbf{w}_{jt}; \beta) \right)' \right],$$

and define $\Psi(\beta)$ as a $S \times D$ matrix,

$$\Psi(\beta) = E \left[\nabla_\beta \mathbf{h}(\mathbf{x}_{jt}; \beta) \right].$$

Finally, define the $(D + S) \times D$ matrix $V(\beta) = [\Phi(\beta); \Psi(\beta)]$, we can now provide conditions for identification of β_0.

Proposition 2. *Suppose there exists an open neighborhood of $\beta_0 \in \Gamma$ in which both $\Phi(\beta)$ and $\Psi(\beta)$ have a constant rank. Then β_0 is locally identifiable if and only if $V(\beta_0)$ has rank D.*[9]

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[9]Local identification is defined in Rothenberg (1971).

Proof. Let the true model is specified as, $r_{jt} = f(\mathbf{w}_{jt}; \beta_0) + u_{jt}$, where,

$$f(\mathbf{w}_{jt}; \beta_0) = \left\{ \ln P_t \left(e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta_0); \eta_0 \right) + \ln \left[e^{\omega_{jt}^*} F(L_{jt}, M_{jt}^*, K_{jt}; \theta_0) \right] \right\}$$

as in (5).

Without loss of generality, assume u_{jt} has normal distribution with mean zero and unit variance.[10] The logarithmic density function of the sample $\{(r_{jt}, \mathbf{w}_{jt})\}_{jt}$ is $-\frac{1}{2} \sum_t [r_{jt} - f(\mathbf{x}_{jt}, \beta_0)]^2$. Thus, for a given β, $\Phi(\beta)$ defined in Proposition 2 is the information matrix. The additional restrictions that are utilized for identification are $E[\mathbf{h}(\mathbf{x}_{jt}, \beta)] = 0$, with Jacobean matrix $\Psi(\beta)$. Thus, our model fits the nonlinear regression framework of Rothenberg (1971) and we can apply Theorem 2 in Rothenberg (1971) to show local identification. □

Komunjer (2012) provides conditions for global identification in the context of non-linear moment equalities models, of which our model is a special case. Identification clearly relies on the structural information provided through firms' first order conditions. In recent work, Gandhi et al. (2013) have established nonparametric identification of production functions when input prices are assumed to be homogeneous. Under heterogenous input prices, it is difficult to recover the unobserved M_{jt} from (1) without a parametric form of the production function. Moreover, the issue of multiple possible materials quantities satisfying (1) becomes more severe, leading to the possibility of partial identification.

With identification conditions established, all parameters can be estimated via GMM:

$$\hat{\beta} = \operatorname{argmin}_\beta \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) \right]' W \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) \right], \tag{7}$$

where $\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) = [\nabla_\beta f(\mathbf{w}_{jt}; \beta)(r_{jt} - f(\mathbf{w}_{jt}; \beta)); \mathbf{h}(\mathbf{x}_{jt}; \beta)]$ and W is a positive semi-definite weight matrix. When the problem is over-identified, we use two-step GMM to obtain the optimal weight matrix. Appendix 1.4 discusses consistency and the asymptotic distribution of this estimator.

1.4   Consistency and Asymptotic Distribution

Now we establish the consistency and asymptotic distribution of our estimator. Since our estimator is an extremum estimator, we need (a) identification; (b) uniform convergence of the objective function for consistency. Since (a) has been established, we focus on (b).

Define

$$\Omega(\beta) = E\Big[\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)\Big]' W E\Big[\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)\Big],$$

and

$$\Omega_n(\beta) = \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)\right]' W \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)\right],$$

where n is the number of observations.

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[10]This assumption is used only to fit the notation and terminology of Rothenberg (1971), if u_t is not normally distributed then $-\frac{1}{2} \sum_{jt} [r_{jt} - f(\mathbf{w}_{jt}, \beta_0)]^2$ is not a logarithmic density. However $\Phi(\beta)$ is still the information matrix of a nonlinear least squares problem and the remainder of the proof is unchanged.

Assume the true parameter $\beta_0 \in \Gamma$, and $\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)$ is continuous in $\beta \in \Gamma$ with probability one. Also, assume $E\left[\sup_\beta |\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)|\right] < \infty$. Then, the Uniform Law of Large Number implies:

$$\sup_\beta \left| \frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) - E\left[\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)\right] \right| = o_p(1).$$

This in turn implies the uniform convergence of the objective function:

$$\sup_\beta \left| \Omega_n(\beta) - \Omega(\beta) \right| = o_p(1).$$

Therefore, the estimator defined for the general model (7) is consistent. For asymptotic distribution of the estimator, define

$$A = E\left[\frac{\partial \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)}{\partial \beta'} \Big|_{\beta = \beta_0} \right],$$

and

$$B = E\left[\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta_0) \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta_0)' \right].$$

These matrices can be estimated by their empirical analogues. If the weight matrix W is a consistent estimator of B^{-1} the asymptotic distribution is,

$$\sqrt{n}(\hat{\beta} - \beta_0) \to N(0, [A'B^{-1}A]^{-1}).$$

Online Appendix 2: Translog Parametric Example

This appendix shows how to implement our method using the translog production function specification as a secondary example to the CES implementation presented in the main paper. The translog production function is specified,

$$Q_{jt} = e^{\omega_{jt}} F(L_{jt}, M_{jt}, K_{jt}; \theta) \tag{8}$$

$$= e^{\omega_{jt}} \exp\left\{ \alpha_k \ln K_{jt} + \alpha_l \ln L_{jt} + \alpha_m \ln M_{jt} + \frac{1}{2}\alpha_{kk} (\ln K_{jt})^2 + \frac{1}{2}\alpha_{ll} (\ln L_{jt})^2 + \frac{1}{2}\alpha_{mm} (\ln M_{jt})^2 \right.$$

$$\left. + \alpha_{kl} (\ln K_{jt})(\ln L_{jt}) + \alpha_{km} (\ln K_{jt})(\ln M_{jt}) + \alpha_{lm} (\ln L_{jt})(\ln M_{jt}) \right\}$$

Where $\theta = (\alpha_k, \alpha_l, \alpha_m, \alpha_{kk}, \alpha_{ll}, \alpha_{mm}, \alpha_{kl}, \alpha_{km}, \alpha_{lm})$ are the parameters to be estimated. The translog is a more flexible generalization of the Cobb-Douglas production function which allows for the elasticity of substitution to be a function of the inputs. We first show that Proposition 1 can be applied so that materials quantities can be recovered. Under this specification,

$$
\begin{aligned}
z(M_{jt}; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) &= \frac{F_{L_{jt}} L_{jt}}{F_{M_{jt}} M_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{\alpha_l + \alpha_{ll} \ln L_{jt} + \alpha_{kl} \ln K_{jt} + \alpha_{ml} \ln M_{jt}}{\alpha_m + \alpha_{mm} \ln M_{jt} + \alpha_{km} \ln K_{jt} + \alpha_{ml} \ln L_{jt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}} \\
&= \frac{S_{Ljt}}{S_{Mjt}} - \frac{E_{L_{jt}}}{E_{M_{jt}}}
\end{aligned}
$$

Where S_{Ljt} and S_{Mjt} are the numerator and denominator of the first term, respectively. The partial derivative of z with respect to M_{jt} is,

$$\frac{\partial z}{\partial M_{jt}} = \frac{1}{M_{jt} S_{Mjt}} \left(\alpha_{ml} - \alpha_{mm} \frac{S_{Ljt}}{S_{Mjt}} \right).$$

So the sign is determined by $\alpha_{ml} - \alpha_{mm} \frac{S_{Ljt}}{S_{Mjt}}$.[11] At any solution where $z(\cdot) = 0$, we know $\frac{S_{Ljt}}{S_{Mjt}} = \frac{E_{Ljt}}{E_{Mjt}}$, so for any M such that $z(M, L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta) = 0$,

$$\text{sign}\left(\frac{\partial z}{\partial M_{jt}} \right) = \text{sign}\left(\alpha_{ml} - \alpha_{mm} \frac{E_{Ljt}}{E_{Mjt}} \right),$$

which does not vary with M_{jt} given $(L_{jt}, K_{jt}, E_{Ljt}, E_{Mjt}; \theta)$. Applying Proposition 1, we can recover M_{jt} as long as $\alpha_{ml} - \alpha_{mm} \frac{E_{Ljt}}{E_{Mjt}} \neq 0$.[12] In this case the closed form for M_{jt}^* is,

$$M^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta) = \tag{9}$$
$$\exp\left(\frac{\frac{E_{Ljt}}{E_{Mjt}} (\alpha_m + \alpha_{km} \ln K_{jt} + \alpha_{ml} \ln L_{jt}) - (\alpha_l + \alpha_{ll} \ln L_{jt} + \alpha_{kl} \ln K_{jt})}{(\alpha_{ml} - \alpha_{mm} \frac{E_{Ljt}}{E_{Mjt}})} \right).$$

Note that for higher order translog specifications, a similar procedure may be available, but it will necessitate finding the roots of a polynomial (rather than linear) equation in M, introducing the possibility that $z(\cdot) = 0$ may have multiple solutions.[13]

To recover the unobserved productivity, we substitute (9) into the labor first order conditions and solve for ω_{jt}^*,

$$\omega_{jt}^* = \frac{1}{1 + 1/\eta} [- \ln(1 + 1/\eta) + \ln P_{Ljt} - \ln F_{Ljt} - 1/\eta \ln F_{jt}]. \tag{10}$$

That is, productivity can be written as a known function of $(L_{jt}, K_{jt}, E_{Ljt}, E_{Mjt})$ up to parameters (θ, η), since $\ln F_{Ljt}$ and $\ln F_{jt}$ are functions of these variables. As with the CES implementation, we assume demand follows a Dixit-Stiglitz specification. Substituting both (9) and (10) into the log revenue function yields our estimating equation,

$$
\begin{aligned}
\ln R_{jt} &= -\ln(1 + 1/\eta) + \ln P_{Ljt} - \ln F_{Ljt} + \ln F_{jt} + u_{jt} \\
&= -\ln\left(1 + \frac{1}{\eta}\right) + \ln\left(E_{Mjt} - \frac{\alpha_{mm}}{\alpha_{ml}} E_{Ljt}\right) \\
&\quad - \ln\left[\left(\alpha_m - \alpha_l \frac{\alpha_{mm}}{\alpha_{ml}}\right) + \left(\alpha_{km} - \alpha_{kl} \frac{\alpha_{mm}}{\alpha_{ml}}\right) \ln K_{jt} \right. \\
&\quad \left. + \left(\alpha_{lm} - \alpha_{ll} \frac{\alpha_{mm}}{\alpha_{ml}}\right) \ln L_{jt} \right] + u_{jt},
\end{aligned}
$$

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[11]We know S_{Mjt} is positive because it is proportional to the product of the marginal product of materials and M_{jt}.

[12]A sufficient condition for $\alpha_{ml} - \frac{E_{Ljt}}{E_{Mjt}} \alpha_{mm} \neq 0$ to always hold is $\alpha_{mm} \alpha_{ml} < 0$.

[13]It would be ideal to be able to recover M^* for a nonparametric production function. Unfortunately, our method requires a parametric approach to recovering M^*, as is illustrated by these examples. Nonetheless, we can in principle accommodate an arbitrarily flexible parametric specification with appropriate restrictions to guarantee uniqueness.

which can be rewritten, in the notation of the general model (7), as

$$r_{jt} = f(\mathbf{w}_{jt}; \beta) + u_{jt}, \tag{11}$$

where β is the vector of parameters, including η and all α's.

As with the CES case in the main paper, not all parameters of the production function are identified by the revenue equation alone. In particular, it is clear that only nonlinear combinations of production and demand parameters are identified from the revenue equation. Moreover, α_k, α_{mk}, and α_{kk}, are canceled when computing $\ln F_{jt} - \ln F_{Ljt}$. Therefore, we need additional restrictions to help identify all production and demand parameters separately, these can be derived from either cross-sectional or time-series assumptions. With the translog specification, cross sectional restrictions like those used in the CES case of the main paper are not easy to find. Instead we follow Olley and Pakes (1996) and Doraszelski and Jaumandreu (2013) in using time series restrictions on productivity to help identify remaining parameters.[14] In particular, we assume that productivity follows a first order Markov process,

$$\omega_{jt} = g(\omega_{jt-1}) + \epsilon_{jt}, \tag{12}$$

where ϵ_{jt}, the productivity innovation, is independent of capital as well as variable labor and material input at time $t-1$. Given β, we can calculate ω_{jt}^* from (10) and estimate $\hat{g}$ from (12).[15] Then, we can define,

$$\epsilon_{jt}(\mathbf{w}_{jt}; \beta) = \omega_{jt}^* - \hat{g}(\omega_{jt-1}^*).$$

By the assumption on the productivity innovation process, $\epsilon_{jt}(\mathbf{w}_{jt}; \beta_0)$ must be uncorrelated with the firms' information set at time $t-1$. This provides additional restrictions that, together with the revenue equation (11), allow us to identify all the parameters. Let $\mathbf{x}_{jt}$ be a vector of instruments which are in the firms information set at time $t-1$, and define $\mathbf{h}(\mathbf{x}_{jt}; \beta) = \epsilon_{jt}(\mathbf{w}_{jt}; \beta)\mathbf{x}_{jt}$. Thus we can construct a set of moment conditions, $E[\mathbf{h}(\mathbf{x}_{jt}; \beta)] = 0$. If the dimension of $\mathbf{x}_{jt}$ is large enough such that $V(\beta)$ has full column rank, all parameters are identified.[16]

Following the general model (7), define the full set of moment conditions as, $E[\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta)] = 0$, where $\mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) = [\nabla_\beta f(\mathbf{w}_{jt}; \beta)(r_{jt} - f(\mathbf{w}_{jt}; \beta)); \epsilon_{jt}(\mathbf{w}_{jt}; \beta)\mathbf{x}_{jt}]$. We estimate all parameters via the following GMM sample analogue,

$$\hat{\beta} = \operatorname{argmin}_\beta \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) \right]' W \left[\frac{1}{n} \sum_{jt} \mathbf{m}(\mathbf{w}_{jt}, \mathbf{x}_{jt}; \beta) \right].$$

To show identification, a similar exercise can be carried out for the translog case as we describe for the CES case in the main paper. Denote

$$\beta = (\eta, \alpha_k, \alpha_l, \alpha_m, \alpha_{kk}, \alpha_{ll}, \alpha_{mm}, \alpha_{kl}, \alpha_{km}, \alpha_{lm}),$$

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[14]From (10), we know that α_k, α_{mk}, and α_{kk} are not cancelled in ω_{jt}^*.

[15]In principle this can be a parametric or non-parametric regression, depending on the assumptions on g. If it is parametric, then we could easily incorporate this estimation into our GMM approach and estimate β and the parameters of g in a single step.

[16]A valid set of instruments could be

$$\mathbf{x}_{jt+1} = \left(\ln\left(\frac{X_{jt}}{\overline{X}}\right), \left(\ln\left(\frac{X_{jt}}{\overline{X}}\right)\right)^2 \right)$$

where $X = L_{jt}, K_{jt}, E_{Mjt}$.

so there are ten parameters to identify. The rank of $\Phi(\beta)$ is four. With additional six moment restrictions as specified in the paper, $\Psi(\beta_0)$ has column rank six (assuming the instruments are not perfectly collinear). Thus, $V(\beta_0)$ has rank ten, and all parameters are locally identified.

Online Appendix 3:   Multiple Materials Inputs

In the main paper, we have followed the literature in assuming that firms purchase a single homogeneous intermediate input. Indeed, the ability to treat the recovered firm-specific price and quantity choices as quality-adjusted scalars representing a single homogenous input is critical since our demand specification assumes that outputs are horizontally differentiated.[17] In reality, intermediate input expenditures are an aggregate of a wide variety of different input goods. Ideally, an analyst would be able to account for each of these goods separately in the production function. Unfortunately, datasets typically contain only the total material expenditure, not information on the various types used, much less prices and quantities for each. With such limited data, it is clearly not possible to learn the impact of individual inputs. In this Appendix, we establish conditions under which production function parameters can be recovered even if the full vector of intermediate input expenditures is not directly observed. The key assumption of this appendix is that the effect of inputs on production can be summarized through a homogeneous materials index function. In the following section, we show how our results extend the case with multiple unobserved materials. We then present the results of a monte carlo study based on this extension.

3.1   Multiple Materials Extension

We first reprise the required assumptions to extend the main results to the case of a vector of multiple unobserved materials, which are briefly discussed in the main paper. Suppose the firm may use up to D_M different types of materials. Denote the vector of material quantities used in production as $M_{jt} = (M_{1jt}, M_{2jt}, \ldots, M_{D_M jt})$. These input types may be entirely different input goods (thread versus fabric) or the same input good of different quality (cotton versus polyester fabric). However, only the total expenditure on all components $E_{M_{jt}} = \sum_{d=1}^{D_M} P_{M_{djt}} M_{djt}$, rather than each specific component M_{djt}, is known to the researcher. Assume inputs enter into the production function as,

$$Q_{jt} = e^{\omega_{jt}} F(L_{jt}, \mu(M_{jt}), K_{jt}; \theta), \tag{13}$$

where $\mu : \mathbb{R}_+^{D_M} \to \mathbb{R}_+$ is a homogeneous index function which summarizes the contribution of all materials inputs to production.[18] As part of the production function, we assume that μ is known to the firm. While this structure allows materials to substitute for each other in an unknown manner, it does restrict the substitution patterns between materials and other production inputs, namely labor and capital. As a result, we can allow for vertically or horizontally differentiated materials and treat them as different elements of the materials vector in M_{jt}. The corresponding idiosyncratic material prices for each component is summarized in price vector $P_{Mjt} = (P_{M1jt}, P_{M2jt}, \ldots, P_{MD_Mjt})$, which is observed by firms but not by researchers.

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[17]We thank an anonymous referee for making this point.

[18]We assume that firms optimally purchase a positive amount of all goods so (15) holds. To accommodate the possibility that some firms do not use some inputs, we can allow for a discrete choice between homogeneous production technologies, e.g., $\mu(M_{jt}) = \max(\mu^1(M_{jt}^1), \mu^2(M_{jt}^2))$ where $M_{jt} = (M_{jt}^1, M_{jt}^2)$ and $\mu^1(\cdot)$ and $\mu^2(\cdot)$ are homogeneous functions of the same degree. Then, only the first order conditions with respect to the profit maximizing technology are relevant. We will use this more general setup in the Monte Carlo experiment in Section 3.2.

The firm's static optimization problem is now to choose L_{jt} and the vector M_{jt} to maximize the profit given productivity and capital stock:

$$\max_{L_{jt}, M_{jt}} P_t(Q_{jt}; \eta) Q_{jt} - P_{Ljt} L_{jt} - P_{M_{jt}} M_{jt}$$

$$\text{s.t. } Q_{jt} = \exp(\omega_{jt}) F(L_{jt}, \mu(M_{jt}), K_{jt}; \theta).$$

The first order conditions for L_{jt} and all components of the vector M_{jt} are:

$$e^{\omega_{jt}} F_{L_{jt}} \left[P_t(Q_{jt}; \eta) + Q_{jt} \frac{\partial P_t(Q_{jt}; \eta)}{\partial Q_{jt}} \right] = P_{Ljt}, \tag{14}$$

$$e^{\omega_{jt}} F_{\mu_{jt}} \left[P_t(Q_{jt}; \eta) + Q_{jt} \frac{\partial P_t(Q_{jt}; \eta)}{\partial Q_{jt}} \right] \mu_d(M_{jt}) = P_{Md_{jt}}, \qquad \forall d = 1, 2, \ldots, D_M \tag{15}$$

where $\mu_d(M_{jt}) = \frac{\partial \mu(M_{jt})}{\partial M_{djt}}$.

Denote the optimal choice of the firm as L_{jt}^* and the vector M_{jt}^*. Thus the total expenditure on materials, which is observed by the researcher, is $E_{M_{jt}}^* = \sum_{d=1}^{D_M} P_{Mdjt} M_{djt}^*$. Define the material price index as $P_{\mu jt} = \frac{E_{M_{jt}}^*}{\psi(M_{jt}^*)}$, where $\psi(M_{jt}^*) = \sum_{d=1}^{D_M} M_{djt}^* \mu_d(M_{jt}^*)$. Using this price index, the information in (15) can be summarized into a single equation by multiplying (15) by M_{djt}^*, summing across d, and dividing it by $\psi(M_{jt}^*)$,

$$e^{\omega_{jt}} F_{\mu_{jt}} \left[P_t(Q_{jt}; \eta) + Q_{jt} \frac{\partial P_t(Q_{jt}; \eta)}{\partial Q_{jt}} \right] = P_{\mu_{jt}}. \tag{16}$$

This equation together with (14) can be viewed as the first order conditions of the firm's optimization problem if it faced labor price $P_{L_{jt}}$ and a material price $P_{\mu_{jt}}$ for *single* material μ. The following proposition shows how our method can be adapted to a production function where the structure of $\mu(\cdot)$ is unknown by the researcher.

Proposition 3. *Suppose the index function $\mu_{jt} = \mu(M_{jt}) : \mathbb{R}_+^{D_M} \to \mathbb{R}_+$ is homogeneous of degree $\kappa > 0$. Then given parameter θ, the firm's optimal choices of input quantities and expenditure satisfy the following equation:*

$$z(\mu_{jt}; L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}, \theta, \kappa) \equiv \frac{F_{L_{jt}} L_{jt}}{F_{\mu_{jt}} \mu_{jt}} - \frac{E_{L_{jt}}}{\frac{1}{\kappa} E_{M_{jt}}} = 0. \tag{17}$$

In addition, this equation admits a unique solution $\mu^*(L_{jt}, K_{jt}, E_{Mjt}, E_{Ljt}; \theta, \kappa)$ when the conditions of Proposition 1 hold for $z(\cdot)$ as defined in (17).

Proof. Note that the firm's optimal choices of input quantities and expenditure satisfy the first order conditions (14) and (16) developed in the main body of the paper. Taking the ratio of the two equations,

$$\frac{F_{L_{jt}}}{F_{\mu_{jt}}} = \frac{P_{L_{jt}}}{P_{\mu_{jt}}},$$

which implies,

$$\frac{F_{L_{jt}} L_{jt}}{F_{\mu_{jt}} \mu_{jt}} = \frac{E_{L_{jt}}}{P_{\mu_{jt}} \mu_{jt}}. \tag{18}$$

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Recall $P_{\mu jt}$ is the material price index which is defined as,

$$P_{\mu jt} = \frac{E_{M_{jt}}}{\sum_{d=1}^{D_M} M_{djt} \mu_d(M_{jt})}.$$

Euler's Theorem for homogeneous functions implies that $\sum_{d=1}^{D_M} M_{djt} \mu_d(M_{jt}) = \kappa \mu(M_{jt})$ given $\mu(\cdot)$ is homogenous of degree κ. Therefore, $P_{\mu jt} = \frac{E_{M_{jt}}}{\kappa \mu(M_{jt})}$. Substituting $P_{\mu jt}$ into (18) yields (17).

Recovering the value of μ uniquely is a direct application of Proposition 1. Note that in the single input case, μ, is just the identity function, which is homogeneous of degree 1. Thus in this special case we know $\kappa = 1$.

Finally, note that extending this proof to the case of a discrete choice between production technologies, as in footnote 18 is straightforward. In this case the index function is $\mu(M) = max_{c \in C}\{\mu^c(M^c)\}$, where c indexes the discrete choice across technologies, all of which are homogeneous of degree κ, and M^c is a sub-vector of M. We need only to replace $\mu(\cdot)$ with the optimally chosen technology, $\mu^c(\cdot)$ in the definition of $P_{\mu jt}$ and apply the same proof using the first order conditions with respect to the sub-vector M^c. Consequently, a firm's materials price-index will be determined by its optimally chosen technology, ignoring those materials types which it does not use as inputs. $\square$

We can now substitute μ^* into the revenue function as with M^*, albeit with κ as an additional scale parameter. In some specifications (e.g., translog), κ may not be separately identified from the production function parameters. In this case κ can be normalized to ones without loss of generality, since it is absorbed in the primary parameters of the production function. In other cases (e.g., CES), κ can be identified through the revenue function, where it represents returns to scale of the materials aggregator index μ.[19] In this case, the estimation procedure still follows our method, except that now we substitute the material expenditure with $\frac{1}{\kappa}E_{M_{jt}}$ and estimate κ as an additional parameter of the production function. For example, in the CES specification of the main paper, we can employ the following revenue equation:[20]

$$\ln R_{jt} = \ln \frac{\eta}{1+\eta} + \ln\left[\frac{1}{\kappa}E_{M_{jt}} + E_{L_{jt}}\left(1 + \frac{\alpha_K}{\alpha_L}\left(\frac{K_{jt}/\overline{K}}{L_{jt}/\overline{L}}\right)^\gamma\right)\right] + u_{jt}.$$

To summarize, our method still works if the effect of material inputs on production can be summarized by a homogeneous materials index function.[21] As we would expect, the functional

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[19]Whether or not κ is identified depends on whether or not the production function already accounts for scale effects on materials independent of other inputs. That is, suppose the production function is $F(L, M, K; \theta)$ and consider the alternative $\tilde{F}(L, M, K; \kappa, \theta) = F(L, M^\kappa, K; \theta)$. In $\tilde{F}(\cdot)$, κ may or may not be identified depending on the form of F. If κ is not identified, normalizing $\kappa = 1$ simply returns the researcher to the original specification. For theoretical reasons, some researchers may still want to impose that $\kappa = 1$ even when it is formally identified. This would be essentially equivalent to assuming constant returns to scale in the materials aggregator μ.

[20]Of course, additional restrictions may still be needed to identify all parameters. We conducted a Monte Carlo experiment to verify this result with $\mu(M_{jt}) = M_{jt}^\kappa$, and our method works very well in recovering the primary parameter θ as well as κ. Results are available upon request.

[21]If $\mu(\cdot)$ is not homogenous, then the "total material expenditure" implied by (16) together with (14) will be $\mu(M_{jt}^*)P_{\mu jt} = \frac{\mu(M_{jt}^*)}{\psi(M_{jt}^*)}E_{M_{jt}^*}$ which is not observable (since generally $\frac{\mu(M_{jt}^*)}{\psi(M_{jt}^*)}$ is not a constant). In this case, information on expenditure alone is insufficient to control for variation in materials inputs even if prices are homogenous across firms.

form of $\mu(\cdot)$ is not identified without more information, but its functional form (indeed, even its dimension) is not needed to recover the other production parameters, θ.

Although we assume that $\mu(\cdot)$ is homogeneous, this still allows a vast set of flexible functional forms that may incorporate both vertically and horizontally differentiated materials inputs.

3.2 Monte Carlo

We now verify the validity of our approach to multiple materials inputs through a monte carlo study. For consistency with the monte carlo in the main paper, we again use the basic CES formulation with the same parameterization, and only emphasize the introduction of multiple materials here. Specifically, in addition to labor and capital stock, firms may choose to use any combination of three components (M_1, M_2, M_3) of materials in production. They enter into the production function through the index function μ,[22]

$$\mu(M_{jt}) = \max\left(\left[(\delta M_{1jt})^{\gamma_1} + M_{3jt}^{\gamma_1} \right]^{1/\gamma_1}, \left[M_{2jt}^{\gamma_2} + M_{3jt}^{\gamma_2} \right]^{1/\gamma_2} \right), \tag{19}$$

where $\gamma_1 = \frac{\sigma_1 - 1}{\sigma_1}$ and $\gamma_2 = \frac{\sigma_2 - 1}{\sigma_2}$. The functional form of μ is observable to the firms but not to the researcher. The experiment's basic structure is inspired by the following scenario:[23] M_1 and M_2 are vertically differentiated versions of the same type of input (e.g., two versions of the same part). The third component is a homogenous material M_3, with idiosyncratic price $P_{M_{3jt}}$. M_1 and M_2 differ in their quality such that the efficiency for M_1 in the production process is $\delta < 1$, while the efficiency for M_2 is normalized to be one. They also differ in regard to their substitutability with M_3 (σ_1 and σ_2 may not be equal). They are produced within competitive industries and their prices, P_{M_1} and P_{M_2}, are offered to all firms.[24] The functional form of $\mu(M_{jt})$ implies that the firm will optimally use either M_1 or M_2 since using both can provide no benefit over only purchasing one. The production function is,

$$Q_{jt} = e^{\omega_{jt}} \overline{Q} \left[\alpha_L \left(\frac{L_{jt}}{\overline{L}} \right)^\gamma + \alpha_M \left(\frac{\mu(M_{jt})}{\overline{\mu}} \right)^\gamma + \alpha_K \left(\frac{K_{jt}}{\overline{K}} \right)^\gamma \right]^{\frac{1}{\gamma}}, \tag{20}$$

where $\gamma = \frac{\sigma - 1}{\sigma}$. The firm observes the price vector, $(P_{L_{jt}}, P_{M_1}, P_{M_2}, P_{M_{3jt}})$, and optimally chooses its vector of inputs, $(L_{jt}, M_{1jt}, M_{2jt}, M_{3jt})$. However, only total materials expenditure $E_M = \sum_{d=1}^{3} P_{M_{djt}} M_{djt}$ is observed by the researcher,[25] who is attempting to recover the production function parameters $(\alpha_L, \alpha_M, \alpha_K, \sigma)$, as well as the distribution of $\mu(M_{jt})$, its price index $P_{\mu jt}$, and productivity ω_{jt}.

Importantly, due to the difference in their elasticities of substitution with M_3, the price of M_3 will affect the optimal decision to employ M_1 or M_2 in production (see Figure A.1). Each firm has a cutoff point $\tilde{P}_{M_{3jt}}$, and the choice of M_1 or M_2 depends on whether it faces a price for M_3 above or below this cutoff. To generate our data, we solve the optimization problem for each firm

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[22]Note that this function a discrete choice between two materials technologies that are homogenous of degree one. As discussed in footnote 18, our method can accommodate this case even though either M_{1jt} or M_{2jt} will be optimally set to zero.

[23]We thank a referee for suggesting a version of this Monte Carlo design.

[24]We have also experimented with allowing these prices to be heterogeneous and have also been able to successfully recover the production function parameters.

[25]She also observes $(E_{L_{jt}}, L_{jt}, K_{jt}, R_{jt})$.

j and period t, and obtain the input demand $(L_{jt}^*, M_{1jt}^*, M_{2jt}^*, M_{3jt}^*)$[26], which is then substituted into the demand function and the production function to calculate the other endogenous variables. Therefore, we have generated the entire data set of firm-level variables for each firm j and period t: $\{\omega_{jt}, K_{jt}, I_{jt}, L_{jt}, M_{jt}, E_{L_{jt}}, E_{M_{jt}}, Q_{jt}, R_{jt}, Q_t, P_t\}$, where $M_{jt} = (M_{1jt}^*, M_{2jt}^*, M_{3jt}^*)$. Then we estimate the model with our method only using data set on $\{K_{jt}, L_{jt}, E_{L_{jt}}, E_{M_{jt}}, R_{jt}, Q_t, P_t\}$.

Table A.1 presents the result for $N = 1000$ replications. As discussed earlier, the form of $\mu(M)$ is not identified, but we find that our method recovers the primary parameters (i.e., all α's, σ and η) very well. Also, the material quantity index and price index can be recovered. In Figure A.2, we compare the recovered material quantity index $\hat{\mu}(M)$ and price index $\hat{P}_\mu$ with the true indexes $\mu(M)$ and P_μ.

Online Appendix 4:   CES Normalization

This appendix illustrates how why the CES function is normalized in the literature, describes our normalization when materials quantities are not directly observed.

4.1   Motivation of Normalization

It has been commonly recognized that the CES production function needs to be normalized to give meaningful identification of its parameters. There is a branch of literature analyzing the importance and the method of normalization, which includes de La Grandville (1989), Klump and de La Grandville (2000), Klump and Preissler (2000), de La Grandville and Solow (2006), and Leon-Ledesma, McAdam and Willman (2010).

The current literature has illustrated the key motivation of the normalization in details for two-factor-input production function (see Brown and de Cani (1963), Klump and Preissler (2000) and Leon-Ledesma, McAdam and Willman (2010)). However, we will work with three-factor-input production function, $Q = F(L, M, K)$. It is defined as a linear homogeneous function in which the elasticity of substitution between any two factors is a constant. The idea and motivation of the standard normalization procedure can be easily extended to our case. To see this, let us follow the literature by stating the definition of elasticity of substitution σ:

$$\begin{cases} \dfrac{\partial \ln(M/L)}{\partial \ln(F_L/F_M)} = \sigma \\ \dfrac{\partial \ln(K/L)}{\partial \ln(F_L/F_K)} = \sigma \end{cases}$$

This definition provides us with a second-order partial differential equation system. Given the assumption of the linear homogenous function, the general solution of the equation system is given by,

$$Q = F(L, M, K) = \lambda_1 [L^\gamma + \lambda_2 M^\gamma + \lambda_3 K^\gamma]^{\frac{1}{\gamma}},$$

where $\gamma = \frac{\sigma-1}{\sigma}$, and λs are three arbitrary constants of integration emerging in the process of solving the differential equation system. One particular functional form used in the literature is

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[26]Either M_{1jt}^* or M_{2jt}^* should be zero because of the substitution between the two.

obtained by taking $\tilde{\alpha}_L = \frac{1}{1+\lambda_2+\lambda_3}$, $\tilde{\alpha}_M = \frac{\lambda_2}{1+\lambda_2+\lambda_3}$, $\tilde{\alpha}_K = 1 - \tilde{\alpha}_L - \tilde{\alpha}_M$ and $C = \lambda_1(1 + \lambda_2 + \lambda_3)^{\frac{1}{\gamma}}$, thus

$$Q = F(L, M, K) = C[\tilde{\alpha}_L L^\gamma + \tilde{\alpha}_M M^\gamma + \tilde{\alpha}_K K^\gamma]^{\frac{1}{\gamma}}.$$

Here $\tilde{\alpha}_L$, $\tilde{\alpha}_M$ and $\tilde{\alpha}_K$ are referred as distribution parameters.[27] However, one can obtain different function forms by taking different specifications for λs. Each of these forms is called a family of CES functions. Examples of different families include ones used in Pitchford (1960), Arrow et al. (1961), and David and van de Klundert (1965). Therefore, as shown in the literature, a common baseline point is needed to compare different families of CES functions whose members are distinguished only by different elasticities of substitution. To this end, one needs to fix baseline point for the level of production (Q_0), factor inputs (L_0, M_0, K_0), and the marginal rates of substitution (μ_{ML0}, μ_{KL0}), which are equal to the price ratios (P_{M_0}/P_{L_0}, P_{K_0}/P_{L_0}) because of the cost minimization.[28] For detailed motivation of normalization, refer to La Grandville (1989) and Leon-Ledesma et al. (2010).

4.2 Standard Normalization Procedure

We follow de La Grandville (1989) and Leon-Ledesma et al. (2010) to illustrate the normalization of the three-factor-input CES function. Given the elasticity of substitution σ, for any baseline point $Z_0 = (L_0, M_0, K_0, Q_0, \mu_{ML0}, \mu_{KL0})$, there are four equations about four parameters that characterize one particular family of CES functions:

$$\tilde{\alpha}_L + \tilde{\alpha}_M + \tilde{\alpha}_K = 1, \tag{21}$$

$$\left(\frac{F_M}{F_L}\right)_0 = \frac{\tilde{\alpha}_M}{\tilde{\alpha}_L}\left(\frac{L_0}{M_0}\right)^{1-\gamma} = \mu_{ML0} \equiv \frac{P_{M_0}}{P_{L_0}}, \tag{22}$$

$$\left(\frac{F_K}{F_L}\right)_0 = \frac{\tilde{\alpha}_K}{\tilde{\alpha}_L}\left(\frac{L_0}{K_0}\right)^{1-\gamma} = \mu_{KL0} \equiv \frac{P_{K_0}}{P_{L_0}}, \tag{23}$$

$$Q_0 = C[\tilde{\alpha}_L L_0^\gamma + \tilde{\alpha}_M M_0^\gamma + \tilde{\alpha}_K K_0^\gamma]^{\frac{1}{\gamma}}. \tag{24}$$

The equations (22) and (23) are implied by cost minimization. Note that (23) implicitly assumes the optimal choice of capital stock in the short run. The last equation holds since Q_0 is the physical output produced by its corresponding factor inputs. De La Grandville (1989) provides a graphical representation of the normalization. He shows that, after normalization all CES functions in the same family share the common baseline point of tangency, although their elasticities of substitution are different. Therefore, the purpose of normalization is to compare different CES functions in a meaningful way: on the one hand, different families of CES functions can be characterized by different baseline points, on the other hand, the members of each family sharing common baseline point are distinguished only by different elasticities of substitution.

These four equations imply a solution for four parameters:

$$\tilde{\alpha}_L(\sigma, Z_0) = \frac{P_{L0}L_0^{\frac{1}{\sigma}}}{P_{M0}M_0^{\frac{1}{\sigma}} + P_{L0}L_0^{\frac{1}{\sigma}} + P_{K0}K_0^{\frac{1}{\sigma}}},$$

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[27]We use $\tilde{\alpha}$'s to denote the un-normalized (or "original") distribution parameters, while α's are reserved for the normalized distribution parameters, unless otherwise noticed.

[28]Note that P_{K_0} is the user price of capital, which usually is not accurately measured. To this end, we will extend the normalization to cases where P_{K_0} is not available.

$$\tilde{\alpha}_M(\sigma, Z_0) = \frac{P_{M0} M_0^{\frac{1}{\sigma}}}{P_{M0} M_0^{\frac{1}{\sigma}} + P_{L0} L_0^{\frac{1}{\sigma}} + P_{K0} K_0^{\frac{1}{\sigma}}},$$

$$\tilde{\alpha}_K(\sigma, Z_0) = \frac{P_{K0} K_0^{\frac{1}{\sigma}}}{P_{M0} M_0^{\frac{1}{\sigma}} + P_{L0} L_0^{\frac{1}{\sigma}} + P_{K0} K_0^{\frac{1}{\sigma}}},$$

$$C(\sigma, Z_0) = Q_0 \left[\frac{P_{L0} L_0^{\frac{1}{\sigma}} + P_{M0} M_0^{\frac{1}{\sigma}} + P_{K0} K_0^{\frac{1}{\sigma}}}{P_{L0} L_0 + P_{M0} M_0 + P_{K0} K_0} \right]^{\frac{\sigma}{\sigma-1}}.$$

Note that given the elasticity of substitution, the value of parameters depend on the choice of baseline point Z_0. Hence, comparing any two CES functions is not informative unless they are specified with the same baseline point.

Substituting the value of these parameters into the original function, we obtain:

$$Q = C(\sigma, Z_0)[\tilde{\alpha}_L(\sigma, Z_0)L^\gamma + \tilde{\alpha}_M(\sigma, Z_0)M^\gamma + \tilde{\alpha}_K(\sigma, Z_0)K^\gamma]^{\frac{1}{\gamma}}.$$

After re-parameterizations, one particular family of CES production function with corresponding normalized parameters is given by

$$Q = Q_0 \left[\alpha_{L0} \left(\frac{L}{L_0} \right)^\gamma + \alpha_{M0} \left(\frac{M}{M_0} \right)^\gamma + \alpha_{K0} \left(\frac{K}{K_0} \right)^\gamma \right]^{\frac{1}{\gamma}},$$

where:

$$\begin{cases} \alpha_{L0} = \dfrac{E_{L_0}}{E_{L_0} + E_{M_0} + E_{K_0}} \\ \alpha_{M0} = \dfrac{E_{M_0}}{E_{L_0} + E_{M_0} + E_{K_0}} \\ \alpha_{K0} = 1 - \alpha_{L0} - \alpha_{M0} \end{cases}$$

and $E_{L_0} = P_{L_0} L_0$, $E_{M_0} = P_{M_0} M_0$ and $E_{K_0} = P_{K_0} K_0$ are expenditures of labor, material and capital respectively.[29] Hence a normalized CES function is characterized by the baseline point Z_0 and elasticity of substitution σ: while each baseline point specifies a family of CES production functions, the members of each family sharing a common baseline values are distinguished only by different elasticities of substitution. The normalized distribution parameters now solely depend on the baseline point. Thus they can be prefixed before the estimation if normalization equations (22)-(23) hold.

4.3 Our CES Normalization

In the standard normalization literature, capital is assumed to be a static input which is chosen optimally in each period. However, in practice, capital may be chosen dynamically. For this reason, we extend the standard normalization approach to allow that capital is not running at the cost-minimizing level in the short run.

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[29]Note that the expenditure on capital E_{K_0} is different from the capital stock K_0. But they are related by $E_{K_0} = P_{K_0} K_0$, where P_{K_0} is the user price of capital stock.

Specifically, although capital could be optimally chosen in the long run, the user price of capital (P_K, if available) may not reflect the marginal cost of capital in the short run. To this end, we assume the choice of capital can deviate from the short-run optimal value by certain magnitude of τ which is treated as a parameter to be estimated. This extension also allows for additional flexibility to deal with situations when the user cost of capital service ($E_K = P_K K$) is not available.

We start from the original production function

$$Q = e^{\tilde{\omega}} F(L, M, K) = e^{\tilde{\omega}} [\tilde{\alpha}_L L^{\gamma} + \tilde{\alpha}_M M^{\gamma} + \tilde{\alpha}_K K^{\gamma}]^{\frac{1}{\gamma}}, \tag{25}$$

where $\tilde{\omega}$ is the firm-level productivity.

As suggested by Leon-Ledesma et al. (2010), the baseline point is chosen as the geometric sample mean:

$$\overline{Z} = (\overline{L}, \overline{M}, \overline{K}, \overline{Q}, \overline{\mu}_{ML}),$$

where $\overline{\mu}_{ML}$ is the average marginal rate of substitution between material and labor (i.e., $\overline{P}_M / \overline{P}_L$).

Note that the choice of the baseline value specifies a family of CES functions. Given the baseline value, the equations that characterize this family are:

$$\tilde{\alpha}_L + \tilde{\alpha}_M + \tilde{\alpha}_K = 1, \tag{26}$$

$$\left(\frac{F_M}{F_L} \right)_{\overline{Z}} = \frac{\tilde{\alpha}_M}{\tilde{\alpha}_L} \left(\frac{\overline{L}}{\overline{M}} \right)^{1-\gamma} = \overline{\mu}_{ML}, \tag{27}$$

$$\left(\frac{F_K}{F_L} \right)_{\overline{Z}} = \frac{\tilde{\alpha}_K}{\tilde{\alpha}_L} \left(\frac{\overline{L}}{\overline{K}} \right)^{1-\gamma} = \left(\tau \frac{\overline{E}_L}{\overline{E}_K} \right) \overline{\mu}_{KL} = \tau \frac{\overline{L}}{\overline{K}}, \tag{28}$$

$$\overline{Q} = e^{\overline{\omega}} [\tilde{\alpha}_L \overline{L}^{\gamma} + \tilde{\alpha}_M \overline{M}^{\gamma} + \tilde{\alpha}_K \overline{K}^{\gamma}]^{\frac{1}{\gamma}}, \tag{29}$$

where $\overline{\omega}$ is the "average" productivity associated with producing $\overline{Q}$ by $(\overline{L}, \overline{M}, \overline{K})$.

Here τ in (28) is introduced as an inefficiency parameter to measure the mean deviation of capital stock from its optimal level. This extension is important for multiple reasons compared with the standard normalization procedure. First, by introducing such an additional flexible parameter, we allow for the case when the capital stock is not optimally chosen in the short run (although it could be optimal in the long run). Specifically, when $\tau = \frac{\overline{E}_K}{\overline{E}_L}$, the marginal rate of substitution of labor and capital at the baseline point is equal to the price ratio, which implies the capital stock is indeed optimally chosen; when $\tau \neq \frac{\overline{E}_K}{\overline{E}_L}$, the actual capital deviates from the optimal amount. We will not specify the value of τ but leave it to be revealed by data as a parameter to estimate. Second, in our empirical application, such a flexible parameter enables us to deal with situations where the average "price" (or the user cost) of capital stock $\overline{P}_K$ (or $\overline{E}_K$) is not available or accurately measured. In other words, instead of assuming that $\overline{P}_K$ or $\overline{E}_K$ is known, we let it be absorbed in the parameter τ which can be estimated from data.

Given γ and τ, the distribution parameters implied by the equations (26), (27) and (28) are

given by:

$$\tilde{\alpha}_L(\gamma, \tau) = \frac{\frac{\overline{E}_L}{\overline{L}^\gamma}}{\frac{\overline{E}_L}{\overline{L}^\gamma} + \frac{\overline{E}_M}{\overline{M}^\gamma} + \tau \frac{\overline{E}_L}{\overline{K}^\gamma}}$$

$$\tilde{\alpha}_M(\gamma, \tau) = \frac{\frac{\overline{E}_M}{\overline{M}^\gamma}}{\frac{\overline{E}_L}{\overline{L}^\gamma} + \frac{\overline{E}_M}{\overline{M}^\gamma} + \tau \frac{\overline{E}_L}{\overline{K}^\gamma}} \qquad (30)$$

$$\tilde{\alpha}_K(\gamma, \tau) = 1 - \tilde{\alpha}_L(\gamma, \tau) - \tilde{\alpha}_M(\gamma, \tau)$$

As in the standard normalization procedure, we plug the distribution parameters into the original CES function to obtain the normalized CES function after re-parametrization:

$$Q = e^\omega \overline{Q} \left[\alpha_L \left(\frac{L}{\overline{L}} \right)^\gamma + \alpha_M \left(\frac{M}{\overline{M}} \right)^\gamma + \alpha_K \left(\frac{K}{\overline{K}} \right)^\gamma \right]^{\frac{1}{\gamma}},$$

where

$$\alpha_L = \frac{\overline{E}_L}{\overline{E}_L + \overline{E}_M + \tau \overline{E}_L}$$

$$\alpha_M = \frac{\overline{E}_M}{\overline{E}_L + \overline{E}_M + \tau \overline{E}_L}, \qquad (31)$$

$$\alpha_K = 1 - \alpha_L - \alpha_M$$

and

$$\omega = \tilde{\omega} - \overline{\omega}.$$

Note that these equations in (31) place restrictions on the value of α's via (31) which is used to help identify all α's as shown in the paper.

Online Appendix 5: Additional Application Results

5.1 Comparison to other estimation methods

While we use the OP-KG estimation method as our primary basis of comparison in the main body of the paper, there are many other approaches to estimating production functions. In this appendix, we compare our method to three additional approaches. First, we implement a simple nonlinear least squares estimator for the production function which proxies for materials with expenditure and also ignores the presence of heterogeneity. Second we use an approach that uses the first order conditions to control for productivity, but continues to use a materials expenditure to proxy for materials quantities. Finally, we compare our estimator to a panel data estimator a la Arellano and Bond (1991), where the productivity term includes a fixed effect and an AR(1) process.

First, we estimate the model with naive nonlinear least square estimation, in which the material expenditure is used as a proxy of quantity and the productivity is lumped into the additive error term. Specifically, the following model is estimated:[30]

$$\ln \left(\frac{R_{jt}}{\overline{R}} \right) = \ln \left(\frac{P_t}{\overline{P}_t} \right) - \frac{1}{\eta} \ln \left(\frac{Q_t}{\overline{Q}_t} \right) + \frac{1}{\gamma} \frac{1+\eta}{\eta} \ln \left[\alpha_L \left(\frac{L_{jt}}{\overline{L}} \right)^\gamma + \alpha_M \left(\frac{E_{M_{jt}}}{\overline{E}_M} \right)^\gamma + \alpha_K \left(\frac{K_{jt}}{\overline{K}} \right)^\gamma \right] + u_{jt}, \qquad (32)$$

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[30]To make results comparable, we estimate this normalized revenue equation instead of production function.

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where $\gamma = \frac{\sigma-1}{\sigma}$ and σ is the elasticity of substitution. P_t and Q_t are industry-level output price and quantity. Note that u_{jt} contains both the productivity and measurement error.

The result is shown in the second column of Table A.2 under the title 'NLLS'. For comparison purposes, the first column reproduces our estimates from Table 3 of the main paper, while the final column reproduces the OP-KG estimates. We can immediately see that controlling for productivity is essential to producing reasonable estimates of the demand parameter η, which has the wrong sign and an extremely high magnitude under the NLLS specification.

Secondly, we estimate the model with nonlinear least square estimation, with the proxy of material quantity (i.e., material expenditure) and the productivity recovered from the first order condition of labor input. To be specific, with the productivity imputed from the first order condition of labor input, the revenue equation can be derived:

$$\ln R_{jt} = \ln \frac{\eta}{1+\eta} + \ln \left[E_{L_{jt}} \frac{\alpha_M}{\alpha_L} \left(\frac{M_{jt}/\overline{M}}{L_{jt}/\overline{L}} \right)^\gamma + E_{L_{jt}} \left(1 + \frac{\alpha_K}{\alpha_L} \left(\frac{K_{jt}/\overline{K}}{L_{jt}/\overline{L}} \right)^\gamma \right) \right] + u_{jt}.$$

Since M_{jt} is not observed, we use its proxy $E_{M_{jt}}$. Thus, the following empirical equation is estimated:

$$\ln R_{jt} = \ln \frac{\eta}{1+\eta} + \ln \left[E_{L_{jt}} \frac{\alpha_M}{\alpha_L} \left(\frac{E_{M_{jt}}/\overline{E}_M}{L_{jt}/\overline{L}} \right)^\gamma + E_{L_{jt}} \left(1 + \frac{\alpha_K}{\alpha_L} \left(\frac{K_{jt}/\overline{K}}{L_{jt}/\overline{L}} \right)^\gamma \right) \right] + u_{jt},$$

with normalization restriction (31).

The result from this table is presented in Column 3 of Table A.2 under the title 'Prod.'. Controlling for productivity generates a reasonable demand parameter, as opposed to the earlier approach. The elasticity of substitution parameter is substantially larger than in all other methods. This is intuitive. Note that the first order conditions for labor and material implies that $E_{L_{jt}} \frac{\alpha_M}{\alpha_L} \left(\frac{M_{jt}/\overline{M}}{L_{jt}/\overline{L}} \right)^\gamma = E_{M_{jt}}$. The difference between this estimation and our method is that we utilize this relationship rather than using a proxy of material quantity. As shown in the table, the elasticity of substitution is significantly larger than our estimates, because the variance of $E_{L_{jt}} \frac{\alpha_M}{\alpha_L} \left(\frac{E_{M_{jt}}/\overline{E}_M}{L_{jt}/\overline{L}} \right)^\gamma$ is larger ($1.2 \sim 3$ times) than the variance of $E_{M_{jt}}$.

Third, we estimate a CES version of persistent panel data method (Arellano and Bond, 1991; Blundell and Bond, 2000). In particular, consider the empirical equation (32), but now the error term is composed of three parts,

$$u_{jt} = \beta_j + \nu_{jt} + \epsilon_{jt},$$

where β_j is a product fixed effect, ϵ_{jt} is an i.i.d. measurement error and ν_{jt} is an i.i.d. innovation term which is assumed to follow an AR(1) process,

$$\nu_{jt} = \rho \nu_{jt-1} + v_{jt}.$$

Following Arellano and Bond (1991) and Blundell and Bond (2000), we take the quasi-difference

$$D_t(\eta, \sigma, \alpha, \rho) = u_{jt} - \rho u_{jt-1},$$

$$D_{t-1}(\eta, \sigma, \alpha, \rho) = u_{jt-1} - \rho u_{jt-2},$$

where $\alpha = (\alpha_L, \alpha_M, \alpha_K)$, and u_{jt} is given by[31]

$$u_{jt} = \ln\left(\frac{R_{jt}}{R}\right) - \left\{ \ln\left(\frac{P_t}{P_t}\right) - \frac{1}{\eta} \ln\left(\frac{Q_t}{Q_t}\right) + \frac{1}{\gamma}\frac{1+\eta}{\eta} \ln\left[\alpha_L \left(\frac{L_{jt}}{\bar{L}}\right)^\gamma + \alpha_M \left(\frac{E_{M_{jt}}}{\bar{E}_M}\right)^\gamma + \alpha_K \left(\frac{K_{jt}}{\bar{K}}\right)^\gamma\right] \right\}.$$

Then we construct the following moment condition for GMM estimation:

$$E\left[D_t(\eta, \sigma, \alpha, \rho) - D_{t-1}(\eta, \sigma, \alpha, \rho)|E_{L_{jt-2}}, E_{M_{jt-2}}, K_{jt-2}, K^2_{jt-2}\right] = 0.$$

The result is reported in the fourth column of A.2 under the title 'AB'. Like the OP-KG estimator, we would expect this the elasticity of substitution to be biased downward using this approach, due to the way the expenditure proxy for materials is employed. In fact, we do see that this estimator, like OP-KG estimates a smaller $\hat{\sigma}$ relative to our method across all four industries.

5.2 Comparing Productivity Measures

As with other structural approaches to production function estimation, there are two potential approaches to defining "productivity" in our model. In the body of the paper, we follow the most common approach, and report the distribution of $\widehat{\omega_{it} + u_{it}}$ which is the residual from the production function itself. This represents the sum of productivity anticipated by the firm as well as unanticipated productivity and potential measurement error in revenues. Alternatively, we could employ (4) to recover $\hat{\omega}_{it}$ alone from the system of first order conditions. This approach includes only an estimate of productivity anticipated by the firm when it makes its labor and materials decision. A similar approach could be recover $\hat{\omega}_{it}$ alone using the OP-KG procedure. However in this case, the anticipated productivity relaters to the firm's expectation of productivity when making the *investment* decision which occurs later than the hiring decision according to the timing assumptions.

It is interesting to see whether these different definitions of the productivity distribution yield substantially different results. We investigate this in Figure A.3, which plots the two distributions for the two different methods for the Clothing industry. (Results for other industries are similar and are available by request). We see that while there is a substantial difference across methods (as is also visible in Figure 3 of the main paper), the difference across definitions for a given method is relatively small. It is particularly small when using our method. This implies that the bulk of the variance in the distribution of productivity is due to anticipated productivity differences, which further supports the importance of controlling for productivity differences when estimating production functions.

Finally, Figure A.4 presents the two distributions for our method only across all four industries. It shows that the result that the two distributions are quite similar is robust across the four industries we consider in the main body of the paper.

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[31]Note that all α's are parameterized as in (31), as a function of τ.

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Figure A.1: Profit difference in choosing different quality levels
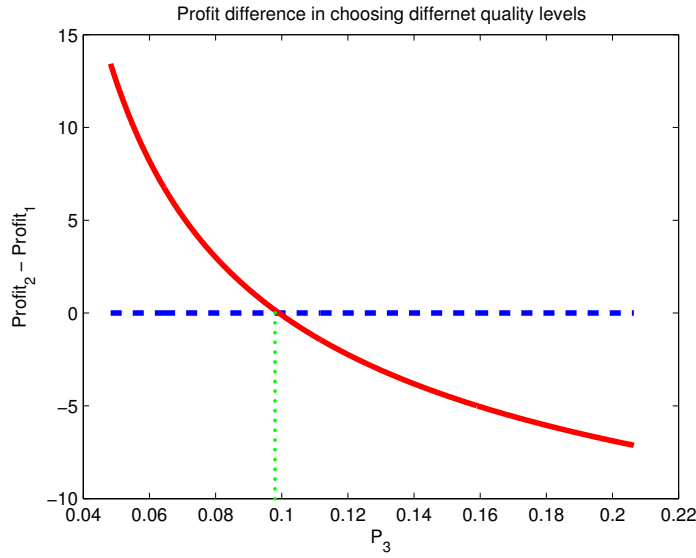
Solid line is the profit difference between choosing $M2$ and $M1$. The cutoff point is demonstrated by the dotted vertical line.

Table A.1: Multiple Materials Monte Carlo: Parameter Estimates[1]

Parameter	$\hat{\eta}$	$\hat{\sigma}$	$\hat{\alpha}_L$	$\hat{\alpha}_M$	$\hat{\alpha}_K$
True	-4.000	1.500	0.400	0.400	0.200
Estimation	-3.997	1.497	0.400	0.400	0.200
SE	(0.001)	(0.026)	(0.004)	(0.005)	(0.008)
RMSE	[0.052]	[0.027]	[0.003]	[0.001]	[0.003]

[1] The table reports the medians of $N = 1000$ replications of each case. The standard errors are included in the parentheses and root mean squared errors are in the square brackets.

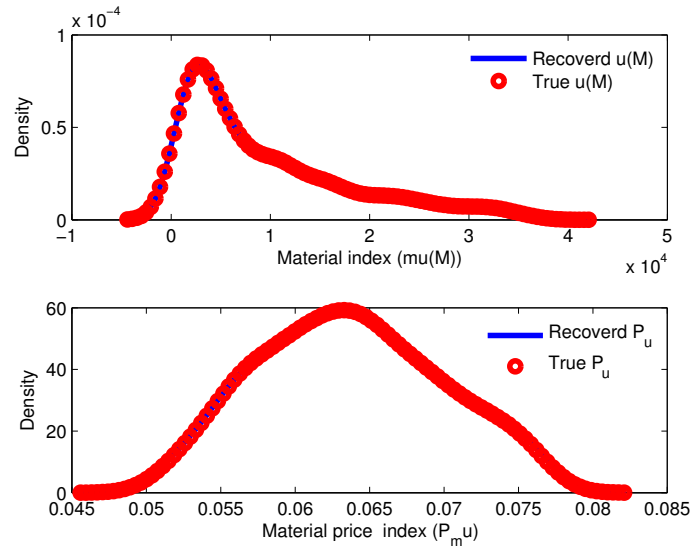Figure A.2: Multiple Materials Monte Carlo: True and recovered material and material price index

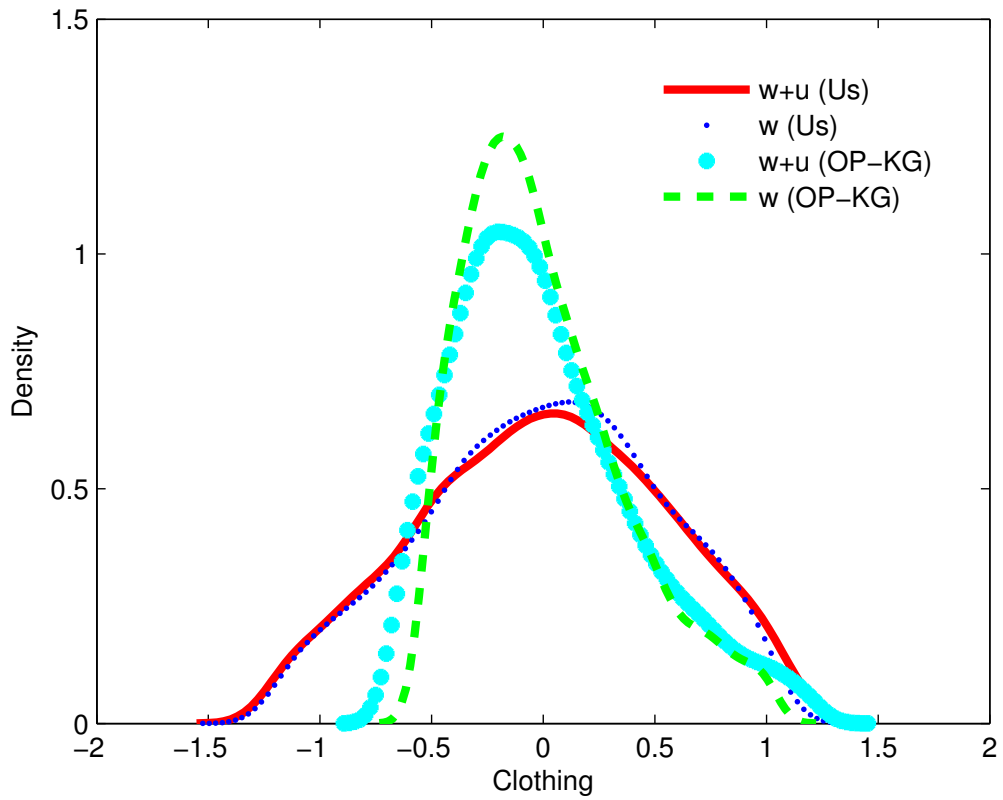Figure A.3: Comparison: densities of $\hat{\omega}$ and $\hat{\omega} + \hat{u}$ – large demonstration

Table A.2: Estimated results by various methods for Colombian industries

	Clothing					Bakery Products				
	Us	NLLS	Prod.	AB	OP-KG	Us	NLLS	Prod.	AB	OP-KG
$\hat{\eta}$	-5.768	109.423	-5.020	-2.416	-8.465	-5.231	24.564	-4.650	-2.546	-5.253
	(0.121)	(39.317)	(0.105)	(0.373)	(1.544)	(0.188)	(1.927)	(0.156)	(0.484)	(0.417)
$\hat{\sigma}$	1.948	1.880	6.826	0.569	0.361	1.443	1.411	4.369	0.363	0.401
	(0.234)	(0.077)	(0.292)	(0.229)	(0.018)	(0.117)	(0.104)	(0.233)	(0.298)	(0.011)
$\hat{\alpha}_L$	0.361	0.372	0.370	0.358	0.371	0.244	0.253	0.254	0.244	0.251
	(0.002)	(0.001)	(0.001)	(0.037)	(0.001)	(0.002)	(0.001)	(0.001)	(0.025)	(0.000)
$\hat{\alpha}_M$	0.601	0.620	0.617	0.597	0.618	0.705	0.731	0.734	0.704	0.725
	(0.003)	(0.002)	(0.002)	(0.062)	(0.001)	(0.006)	(0.002)	(0.004)	(0.072)	(0.001)
$\hat{\alpha}_K$	0.038	0.008	0.013	0.045	0.011	0.050	0.016	0.012	0.053	0.025
	(0.004)	(0.004)	(0.003)	(0.100)	(0.002)	(0.007)	(0.003)	(0.005)	(0.097)	(0.002)
$\hat{g}_0$	0.008		0.048		0.101	0.039		0.029		0.148
	(0.010)		(0.010)		(0.011)	(0.015)		(0.015)		(0.011)
$\hat{g}_1$	0.695		0.771		0.972	0.822		0.873		0.955
	(0.014)		(0.018)		(0.010)	(0.012)		(0.022)		(0.005)
$\hat{\rho}$				0.547					0.264	
				(0.071)					(0.191)	
#Obs	5763					2269				
	Printing & Publishing					Metal Furniture				
	Us	NLLS	Prod.	AB	OP-KG	Us	NLLS	Prod.	AB	OP-KG
$\hat{\eta}$	-4.659	14.142	-3.857	-2.189	-12.161	-5.518	10.514	-4.186	-6.280	-6.947
	(0.236)	(0.986)	(0.135)	(1.739)	(5.434)	(0.433)	(0.918)	(0.210)	(6.713)	(2.686)
$\hat{\sigma}$	2.555	2.218	3.326	0.313	0.593	1.772	1.395	5.600	0.429	0.393
	(0.405)	(0.186)	(0.145)	(0.405)	(0.054)	(0.379)	(0.090)	(0.598)	(0.212)	(0.045)
$\hat{\alpha}_L$	0.372	0.379	0.389	0.298	0.381	0.300	0.295	0.319	0.233	0.304
	(0.005)	(0.003)	(0.004)	(0.070)	(0.004)	(0.005)	(0.002)	(0.003)	(0.057	(0.005
$\hat{\alpha}_M$	0.537	0.546	0.560	0.429	0.549	0.637	0.627	0.677	0.494	0.647
	(0.007)	(0.004)	(0.005)	(0.101)	(0.005)	(0.010)	(0.005)	(0.005)	(0.121)	(0.011)
$\hat{\alpha}_K$	0.091	0.075	0.051	0.273	0.070	0.064	0.078	0.004	0.273	0.049
	(0.013)	(0.006)	(0.009)	(0.171)	(0.009)	(0.015)	(0.008)	(0.008)	(0.178)	(0.016)
$\hat{g}_0$	-0.025		-0.018		0.211	-0.033		-0.003		0.219
	(0.015)		(0.015)		(0.039)	(0.024)		(0.024)		(0.072)
$\hat{g}_1$	0.906		0.933		0.950	0.824		0.863		0.877
	(0.019)		(0.017)		(0.014)	(0.026)		(0.033)		(0.026)
$\hat{\rho}$				0.358					-0.303	
				(0.380)					(0.462)	
#Obs	2377					903				

Figure A.4: Comparison: densities of $\hat{\omega}$ and $\hat{\omega} + \hat{u}$ from our method – small figures